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THE ARITHMETIC TEACHER

a journal of
The National Council of Teachers of Mathematics

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# THE ARITHMETIC TEACHER

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Arithmetic for Today's Six- and Seven-Year-Olds

Agnes G. Gunderson
Moorhead, Minn.

Arithmetic for today's six and seven-year-olds is made up of activities as well as commercially prepared materials (workbooks and textbooks); such activities as playing store with real or toy money; constructing a grocery or post-office; making number charts and arithmetic scrapbooks for the reading table; measuring objects in the schoolroom; measuring each other's heights and weights; activities centering around a unit on time-telling; a unit on money; a unit on measures; and many others. With such a varied content, arithmetic should always be interesting and challenging to our children from the first day of school.

In this report I shall not take up workbooks or textbooks except to say that it is most difficult, if not impossible, to carry out a well-rounded and complete arithmetic program for young children without a good workbook. Here we find arithmetic content arranged in a systematic way with provision for practice and maintenance; functional illustrations also aid the child in understanding arithmetic. A good workbook with accompanying teacher's guide is a boon to the busy primary teacher.

Workbooks represent the *semiconcrete* stage in arithmetic learning. Preceding this is the *concrete* stage wherein the child, through the use of objects or concrete materials, is given opportunities for discovering arithmetic ideas or concepts. It is this specific kind of arithmetic that the beginning teacher finds so difficult, this first stage of number learning, namely, the concrete stage involving the use of objects to discover the ideas and meaning of arithmetic, and leading from this concrete stage through the semiconcrete stage in which the child uses dots, lines, or geometric designs, into the abstract in which number symbols are used to represent quantities.

Arithmetic Today

Some terms that characterize the arithmetic of today are:

1. Discovery by the children, not demonstration by the teacher
2. All children handling materials, not only the teacher
3. Questioning by the teacher, not telling
4. Meaning, not drill (memorizing)
5. Problem solving, preceding, not following, the teaching of number facts
6. Relationships are emphasized, not stimulus response
7. Readiness for multiplication and division taught as early as Grade I and Grade II
8. Fractions (concept, not symbol) introduced as early as Grade I and Grade II
9. Making original problems

Illustrating the first characteristic: discovery by the children, not demonstration by the teacher, includes two other char-

acteristics also, questioning by the teacher not telling, and all children handling materials not only the teacher. Let us take as an example the addition fact $3+2=5$. Some of us can recall how the teacher would demonstrate the addition fact while the children sat passively by. The teacher would hold 3 pencils in one hand asking, "How many pencils do I have in this hand?" Then showing 2 pencils in the other hand, "How many do I have in this hand? Now (moving the two hands together) how many do I have in all?" Following such demonstration the children were expected to memorize the addition fact and use it in solving problems. Contrast that with a method that fosters discovery, wherein the children have had many experiences in grouping objects, and we might add here, that at times the entire arithmetic period of 20 or 25 minutes may be used in working with concrete materials. Each child has 10–15 counters. The teacher may give directions such as: "Working with 6 counters, see how many different patterns you can make. When you have made a pattern with your counters, you may show it on the board." Usually these patterns are shown: the 6 counters in a row, vertical and horizontal; 6 counters forming a circle; and the 6 counters arranged close together in one group. Encouraging further discovery the teacher asks for patterns that are different until all the different combinations: 3 and 3; 2 and 4; 5 and 1; have been discovered. Similar procedure is used with 8 counters, 7, 9, and 10. Children also solve problems in addition and subtraction using counters at this stage. To illustrate, the teacher may say, "Use counters to work this problem:

Mary has 3 blue pencils.
She has 2 red pencils.
How many pencils does she have in all?"

Because we want children to become familiar with the form in which they will later use the number fact, we encourage them to place one addend below the other, whether numbers or counters. The teacher remarks, "I like the way Dick has arranged his counters." Dick shows his arrangement on the board and explains it:

0 0 0	(three pencils)	She had 5 pencils
0 0	(two pencils)	in all.

Later, after many such lessons and problems, the number symbols will be added to this diagram when in response to the teacher's questioning: "Is there any other way we can show 3 pencils and 2 pencils?" the children answer, "You can write 3 pencils and 2 pencils." This is the time to emphasize the various ways of showing or illustrating a problem. The teacher sees the three stages of number learning—the *concrete*, which is represented by objects or counters; the *semiconcrete*, which is the drawing of lines or circles on the board, and the *abstract*, which is number symbols:

concrete	*semiconcrete*	*abstract*	
objects,	0 0 0	3	(pencils)
counters	0 0	2	(pencils).

From here it is but a step to writing the sum and the number fact is completed. It has been discovered in stages; the children found out by counting that 3 and 2 are 5—this is the number fact; the last step, that of recording or completing the algorism, is writing the sum. Contrast this method which is *discovery* by the child with the early arithmetic teaching when the child was faced with the algorism

$$\begin{array}{r} 3 \\ +2 \\ \hline 5 \end{array}$$

and expected to learn it with only the meager demonstration by the teacher of its meaning.

When we say "Meaning—not drill" is characteristic of today's arithmetic, we do not eliminate drill, but drill must be preceded and accompanied by meaning. Motivated practice and use of number facts in problem solving and in playing store will reduce the amount of drill formerly required.

Sharing or division experiences occur frequently in the daily lives of six and seven-year-olds. We hear remarks such as, "Give your sister half," or the child

demanding: "Give me some," and the generous: "I'll give you half of mine." Children find counters helpful in discovering multiplication and division. The following lesson shows how a skillful teacher can challenge and hold the attention of young children. It is also an example of how insightful questioning can lead a child to discover new number ideas: "You may work with 6 counters now. See how many groups you can make with 2 counters in each group. How many counters do you have in each group? How many groups do you have? How many *2's* do you have? Now put the 6 counters in one group again. This time put $\frac{1}{3}$ counters in a group. How many groups did you get? How many *3's* do you have?" Note that this step is not followed by the question: How many are two 3's? Be content to go slowly. The teacher continues: "I'm going to give you something harder this time. Arrange your 6 counters in 2 groups so that there will be the same number in each group." The less mature child will do this by repeated subtractions, "1 in this group; 1 in this group," and so on until the group of 6 has been divided into 2 groups with 3 in each group; while the more mature child may see at once 3 and 3. In similar manner, groups of 8, 4, and 10 counters may be divided into 2 equal groups. It is well to give children experience with uneven division at this time also. When children have divided groups of 6 and of 8 counters into 2 equal groups, the teacher says, "Work with 7 counters this time. Arrange them into 2 groups with the same number in each group, just as you did with the 6 and the 8 counters. Some children will give up easily and say, "I can't do it," but there are usually some youngsters, who at the teacher's urging to "try," persevere and arrive at the conclusion that it is impossible and cannot be done (except by breaking 1 in half). How much more thrilling it is for a child to *discover* that it is impossible to divide 7 counters into 2 equal groups than it is to have the teacher tell him that 7 is an uneven number and can not be divided into 2 equal parts. The next step may be to arrange the 7 counters into 2 groups so that there will be 1 more counter in one group than in the other. Then working with an even number of counters, say 8, the children may be asked to arrange them into 2 groups so that there will be 1 more counter in one group than in the other. This also turns out to be an impossible task. Another interesting discovery is also made here: transferring 1 counter from one of the two equal groups to the other group, increases that group over the other by *2*, e.g. Having 2 groups with 4 counters in each group, then transferring 1 counter from one group to the other leaves 3 counters in one group and 5 in the other. Working with different numbers—even and uneven—children are able to generalize that even numbers can be divided into 2 equal groups, but with uneven numbers there will be a remainder. Children enjoy such challenges and frequently say, "Give us something hard; Give us something impossible to do"; their enthusiasm in trying out this "trick" on their older brothers and sisters shows that to them Arithmetic is fun.

From such grouping of counters it is but a step to solving problems, and just as problems in addition and subtraction should precede the teaching of those facts. so should multiplication and division problems precede the teaching of those facts. Solving problems is motivation for mastery of number facts.

In a study conducted at the University of Wyoming Elementary School, it was found that seven-year-olds who had had much experience in grouping counters, and also solving addition and subtraction problems with counters, could solve correctly simple multiplication problems by addition, and division problems by repeated subtractions. To illustrate, take this simple problem:

What will 4 pencils cost at 3 cents each?

Using counters or drawing lines children

showed four 3's, then counted to get the answer. A few looked at the layout of counters and counted by 3's

Counters (concrete)	*Drawing* (semi-concrete)	*Numbers* (abstract)	
0 0 0	/ / /	3	(The next step is the multiplication fact $\begin{array}{r}3\\ \times 4\\ \hline\end{array}$ taught in Grade Three)
0 0 0	/ / /	3	
0 0 0	/ / /	3	
0 0 0	/ / /	3	

Or take a problem in division:

One candy bar costs 5¢.
How many candy bars can I buy for 15¢?

Some children worked with counters, separated groups of 5 and arranged them in rows:

0 0 0 0 0
0 0 0 0 0
0 0 0 0 0

Some worked on a more mature level, solving the problem mentally, giving as explanation for the answer "3," "5, 10, 15." One child said, "There are three 5's in 15—*5* and *5* and *5*." Taking another division problem:

How many 3-cent stamps can I buy for 17 cents?

which is really a problem in division with a remainder, but easily solved by seven-year-olds who have had experience in grouping. The 24 second graders working individually arrived at the correct solution: 5 stamps and 2 cents left. In no instance was the child confused as to the correct naming of the remainder. The 17 counters were separated into groups of 3 and arranged in rows:

0 0 0
0 0 0
0 0 0
0 0 0
0 0 0
0 0

One child remarked, "If you had one more penny you could buy 6 stamps." This early exposure to uneven division *with concrete materials* should eliminate much of the confusion in the child's mind about remainders in division which we often see in later grades when division as a process is presented. If children are to understand and appreciate multiplication as a short cut in adding like numbers, they must have had many experiences in solving multiplication problems by addition; and in the same way, they must have worked many division problems by subtraction to fully appreciate division. Note that in this early work, the terms: *multiply, multiplication,* and *division* are not used—only the word *divide,* and that only as it occurs in natural conversation; introducing such words at this time would only be a stumbling block to the child and perhaps deter him in solving the problems with the equipment he already has (a knowledge of addition and subtraction).

An understanding of fractions, such as one-half, one-fourth, and one-third is not too difficult for six and seven-year-olds if objects are used. Each child should be provided with objects which can be cut or folded into fractional parts. Paper plates to represent pies are often used; also circles, squares, and rectangles of paper which can be folded into halves and fourths. With papers marked into fourths, halves, and thirds, children have a good time as well as gain knowledge in such a lesson as this: "Take your paper that is marked into fourths; color 1 fourth red; color 1 fourth blue. How many fourths are colored? Leave 1 fourth uncolored, but color the rest of the paper any color you wish. What part will be the color you choose? Take the paper that is marked into thirds. Color 1 third green. Is more or less than one-half of the paper colored green?"

A flannel board is very helpful in teaching fractions, as children can place parts of circles, such as halves, or thirds, or fourths over a whole circle to show how many of the parts are needed to make a whole circle. To many children the flannel board is as much fun as any game. One seven-year-old, working by himself, was intrigued by the discovery that one whole could be made up of 2 thirds and 2 of the

sixths in place of the other third. How much more meaningful the addition of fractions becomes after discoveries such as this.

A student teacher working with a group of 8 six-year-olds showed them a candy bar and said, "Watch, and tell how I divide this candy bar." As she cut it into halves, one child said, "You cut it in half." The lesson continued: "How much is this part? (one half) How much is this other part? (one half) How many people can have one-half of the candy bar? (two) Now watch as I divide another candy bar. Into how many parts has this candy bar been divided? (four) What is each part called? (one fourth) Point to a part that is one half of a candy bar. Point to a part that is one fourth of a candy bar. Which part is larger? How many children are there in our group? (8). Is there a piece of candy for each one?" Various comments were heard; "You can make some of the pieces smaller, I could give part of mine to some one else." The teacher continues, "I'm going to give some of the candy away. Tell me what part of the candy bar I give you." As the fourths were given, each child responded "one fourth." "Now do we have enough pieces left to give each of you one?" As the children looked at the remaining candy bar divided into halves, one child said, "If you cut each part in half, there will be 1 fourth for all," (meaning 1 fourth for each of them). Is not this meaningful teaching? Try something similar with older children who have difficulty in understanding fractions; with several apples of the same size, some divided into thirds, then into sixths; some divided into halves, then into fourths, then into eighths, let children discover relationships between fractions, not only unit fractions, but also such fractions as 3 fourths and 7 eighths, 2 thirds and 5 sixths, and so on.

Making original problems is a part of the arithmetic for today's six and seven-year-olds. At first they will give problems orally, usually from their own buying or sharing experiences; later they will write and illustrate them. These problems then are collected and made into scrapbooks for the reading table, or each child may make a scrapbook of his own and take it home. When writing original problems, time is given for children to read them to the class and call on some one to tell the answer. Occasionally problems have a note of humor which is much appreciated by the class and bring forth remarks as "That's a good one."

Problems by Children

The following problems were written and illustrated by seven-year-olds in the Elementary School of the University of Wyoming.

1. Six children were playing in the yard.
   Two decided to go home.
   How many were left to play?
   Virginia

   ø ø 0 0 0 0

2. I had 6 cents.
   I spent 3 cents.
   How much do I have now?
   Bonnie

   ø ø ø 0 0 0

   $$\begin{array}{r} 6 \\ -3 \\ \hline 3 \end{array}$$

3. I had 16 bunnies.
   6 of the poor things died.
   How many were left?
   Lance

   ø ø ø
   ø ø ø 0 0
   0 0 0 0
   0 0 0 0

4. Ricky had 52 cents allowance.
   I gave him 2 cents.
   How many cents does he have now?
   John

   $$\begin{array}{r} 52 \\ +\ 2 \\ \hline 54 \end{array}$$

5. I had 4 cents.
   I got 6 cents from my mother.
   I spent 3 cents.
   How many cents do I have left?
   Michael

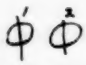
   $$\begin{array}{r} 4 \\ +\ 6 \\ \hline 10 \end{array} \qquad \begin{array}{r} 10 \\ -\ 3 \\ \hline 7 \end{array}$$

   (2-step problems had not been taught)

6. I had 80 cents
I wanted to buy some perfume that cost 40 cents for 1 bottle.
How much money would I have left?
Judy

$$\begin{array}{r} 80¢ \\ -40¢ \\ \hline 40¢ \end{array}$$

7. I had two tablets.
I used one tablet.
Well! Well! I used another tablet.
How many do I have now?
Janna



$$\begin{array}{r} 2 \\ -2 \\ \hline 0 \end{array}$$

8. I had one peach.
I cut it in half.
I gave one half to my brother.
How much do I have left?
Jo Ann

 one half

9. Mother gave me a pear.
I ate half of my pear.
How much do I have left?
Denice

 $\frac{1}{2}$ (Fraction symbols had not been taught)

10. Mother gave me a banana.
I ate one fourth of it.
How much do I have left?
Alfred

 3 fourths

11. I had 2/4 of a pie.
I ate 1/4.
How many fourths do I have left?
Bobby

$$\begin{array}{r} \frac{2}{4} \\ -\frac{1}{4} \\ \hline \frac{1}{4} \end{array}$$

(Fraction symbols had not been taught)

12. I had 2 oranges.
I cut one into halves and one into fourths.
How many halves and how many fourths do I have?
Terry

2 halves 4 fourths
(No computation shown on paper)

13. It was 3:00 o'clock in the morning.
How many hours past midnight? 3.

(No computations)

14. School starts at 9:00 o'clock.
I got there at 20 until 9:00.
How much time did I have to play? 20 minutes

(No computation)

15. I had 20 children come to my party.
12 children had to go home at 4:00 o'clock.
before the party was over.
How many children could stay? 8.

Some children may write more advanced problems as this one of Robert's, which according to textbooks would be classed as a problem involving borrowing in two places:

I had \$10.
I spent 20¢.
How much do I have left? \$9.80

Neither borrowing nor subtraction of 3-place numbers is taught in our second grade but this youngster knows money values. When asked how he worked the problem, he gave this explanation, "If I had spent \$1, I would have had \$9 left; and 20 from \$1 is 80, so I would have \$9.80 left." (there was no computation on the paper). Another problem involving money was this one:

I bought 3 pens for \$1.00, and a toy airplane for 25¢.
I paid with 4 pieces of money.
What were they?

| \$1.00 | 10 | 10 | 5 |

On his paper he had drawn four rectangles and marked them \$1.00, 10, 10, 5.

Arithmetic is one of the most challenging of school subjects and as teachers we should help children see it that way. It is interesting to see how even seven-year-olds make use of the knowledge they have when solving problems which call for computational skills far in advance of what they have been taught. Over a period of several years we have noted that a few of our second-graders solve correctly the most advanced problems given in the Metro-

politan Achievement Test for Grade II. Children who had the correct answers were asked to tell how they had worked the problems. They had evidently solved them mentally as only a very few had any marks or clues on the paper. The problems and methods of solution are as follows:

PROBLEM No. 15.

Three children are coming to my house. I am going to give each one of them 3 cookies. I must have how many cookies for all of them?

Thirteen had the correct answer (9).
Five said, "3 and 3 and 3 are 9."
Four counted, 3, 6, 9.
Two said, "three 3's are 9."
One said, "3 times 3 are 9."

Only one child resorted to drawings or semiconcrete materials. The others handled numbers in the abstract.

PROBLEM No. 16.

Mother gave me 30 cents. I spent 5 cents for a sandwich and 10 cents for a ride on the bus. How many cents did I have left?

Eleven children had the correct answer (15¢)
Five added 10 and 5 to find out what was spent, then said they would have 15¢ left because 15 and 15 are 30.
Three made two subtractions: $30-5=25$; $25-10=15$.
Two made drawings to find the answer.
One child reasoned: 15 and 15 are 30; 30 take away 15 is 15.

PROBLEM No. 17.

An airplane went 90 miles the first hour and 105 miles the second hour. How far did it fly in 2 hours?

Six children had the correct answer (195)
Five said they took the 5 off the 105, then added 100 and 90, then put the 5 on again.
One said he took the 5 off the 105 and put it on the 90, that would be 95; then added 100 and 95.
Another made a very good estimate but did not have the exact answer—"90 and 105 should be about 200."

Roundabout ways—true—but the children used only the numbers pertinent to the solution; the 2 hours given in the problem did not confuse or bewilder them and in spite of detours they reached the goal.

Summary or Conclusions

Discovery is the key word in today's arithmetic. Discovery should not be interpreted as readiness. As Dr. Ben Sueltz said at the meeting of the NCTM in Boston, *Discovery is arithmetic.*

We must consider individual differences among children. There is no one pattern for all children to strive for. Children in a class vary in insight. It is the responsibility of the teacher to challenge each child; to keep him working at his optimum level of achievement; to encourage him to advance into the stage where he can work with abstract numbers, yet make him feel free to use the concrete and semiconcrete materials when he needs them.

The skillful teacher questions more and tells less; she must listen while the child tells what he has discovered—perhaps the reverse of what she herself experienced in the primary grades when the teacher did the telling, explaining, and demonstrating while the child watched and listened. Learning takes time; the child must not be hurried in his discoveries; we must give him time as well as opportunity to tell what he thinks and to explain how he figured it out.

EDITOR'S NOTE. This article by Miss Gunderson is an outgrowth of her presentation at the meeting of the National Council of Teachers of Mathematics in Boston, April, 1955. It is an excellent exposition of modern methods of learning arithmetic applied to younger children. Miss Gunderson has the experience and insight which are so necessary to the "artful questioning" which she so well illustrates. She points out the need for pupil experience and participation at all stages of learning from initial discovery through practice at the symbolic stage. As teachers, we should not be distressed with the seemingly cumbersome methods which children frequently employ: they are discoverers, they are learning, they will finally arrive at the conventional methods used by adults but they will then have appreciation and understanding. One of the readers of Miss Gunderson's manuscript remarked, "I wish I had learned arithmetic from her."

Implementing a Mathematics Program

Arthur Hughson
Assistant Superintendent, New York City

No program of education can be more effective than the means, methods and personnel selected to perform the implementation. Despite all claims to the contrary, teachers still find it easier to teach as they were taught than to teach the way they were (are) taught to teach. The old program in any area of the curriculum must first be held up for inspection before the eyes of those who do the teaching and supervising, its problems must be analyzed and pointed up, and the need for change indicated and emphasized through precept and example. The job of the researcher and administrator becomes that of selling the new because it holds more values than the old in child understandings, learnings and skills and makes possible easier assimilation by pupils. Once teachers and supervisors can see these values, the new program is launched.

The administrator of the program will find many problems, both foreseen and unforseen, as he progresses in his task; problems of creating new materials, problems of selecting the personnel to do teacher education and problems of constant evaluation and re-evaluation as the project proceeds.

For a number of years, we in New York City knew that our children were not doing well in elementary school mathematics. Too many of our bright children looked at the subject as a meaningless chore or even as a *bete noir* while the slower pupils were obviously weighted down in the morass of despair with no hope of getting any help toward mastery of simple manipulation and understanding in the area of mathematics.

In the fall of 1947, a tentative, new program designated "Developmental Mathematics" was organized experimentally on a city-wide basis. Twenty-three teachers who had been assigned to "Remedial Arithmetic" were reoriented toward this new program. This corps was selected by each of the Assistant (field) Superintendents[1] to spread the gospel of meaningful mathematics. After some time it became evident to the administration of the program that not only the teachers who were actually to do the teaching in the classrooms but the supervisors, too, would have to become familiar with the new philosophy. Courses and Workshops were organized for supervisors[2] (and for teachers). A number of principals who had experimented in this area in one district agreed to give a series of three three-hour sessions on meaningful or Developmental Mathematics to their colleagues in other districts.

One problem that resulted from the necessity of producing teaching materials was met by introducing the program slowly—in one grade—each year. The program of production and implementation of materials was developed cooperatively by the Bureau of Curriculum Research and the Division of Elementary Schools. Implementation began with Grade I in 1948–49, and reached Grade VI in 1953–54. Even so there were times and occasions when we had to backtrack to orient new teachers appointed to various grades. Also, certain emphases which were necessary in the beginning of the project were modified as teachers and supervisors came to understand meaningful mathematics and as they developed procedures in harmony with the changing needs of

[1] There were 23 such areas. At present there are 25.

[2] Such courses, etc. had been organized for teachers before this.

children. For example, at first great emphasis was placed on the use of concrete, illustrative materials. (The failure to use these as aids to mathematical understanding was one of the criticisms of the old program.) As time went on we learned that it was necessary for teachers to learn more mathematics in order to help the pupils use concrete aids more effectively. Thus, too, in the beginning, we found it necessary to minimize in the first grade the teaching of number and hold off paper and pencil manipulation until well into the second year of the child's school life.[3] As teachers became more adept we found that pupils could move somewhat faster without any handicap.

The mathematics program was based on research. Evenutally, we came to accept action research as an important step in curriculum development. Our course of study is the result of curriculum research and teacher and child experience and reaction. We feel today that nothing should be accepted as desirable curriculum or methodology which has not first been tested by learning (by the children) and teaching (by the teachers).

The writer regrets that in order to cover the field many of the adopted activities are just barely mentioned and many of the problems are not even indicated. All this because only sketchy treatment can be given in a brief article to all those pressing problems that confronted us from day to day during the years of trial and experimentation. Some day the complete experiment will probably be written up. Only then will we have a picture of the durnal problems that had to be solved in launching the experiment, keeping it on even keel and bringing it safely into the harbor as a legitimate and desirable portion of our curriculum.

[3] Children were admitted to the first grade at 5 yrs. 4 mos. at this time.

Research Award not Granted

At the time of the 1952 Christmas meeting of the National Council of Teachers of Mathematics, the Board of Directors approved a recommendation of the Research Committee of the Council that a research award be made in the form of a $1,000 scholarship to be awarded to the individual who has a masters degree and who submits the best prospectus for a research study in learning problems in the field of mathematics, including arithmetic.

Notices of this action were sent to colleges and universities and an announcement was published in the October, 1953 issue of *The Mathematics Teacher*. Certain basic conditions needed to be satisfied and a prospectus was to be submitted before January 1, 1955. The award was to be made at the annual meeting of the Council in 1955 if a prospectus deemed worthy of being so honored was received.

Prior to January 1, 1955, a prospectus had been received from each of six people. The Board of Directors then appointed a panel of judges to review the material received and make recommendations concerning an award. The panel consisted of Zeke Loflin, Henry Van Engen, and Bruce Meserve, Chairman.

This panel reported that after careful consideration of the manuscripts submitted it was recommended that no award be made this year. Some of the manuscripts could not be considered because they did not comply with the conditions of the award.

The committee made several suggestions with respect to other award possibilities. As a result of these suggestions the Board of Directors requested the Research Committee to reconsider the matter of awards and the encouragement of research and make more specific recommendations to the Board at a later date.

News about future research awards will appear in this JOURNAL.

The Role of Experiences in Arithmetic

Goldie Nadelman and Elsie B. Paskins*
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In every modern elementary school, there are many natural situations which provide opportunities for children to think about and to use numbers. Classroom and school experiences are important because they pave the way for planning a program of meaningful mathematics. Teachers welcome suggestions which help them in recognizing and utilizing experiences effectively, in setting up situations which create a stimulating learning environment, and in stimulating children to play an active part in the learning activities.

The following typical classroom experience is rich not only in opportunity for emphasizing mathematical meanings but also in social significance and in practice for children to solve everyday problems that concern them. Arithmetic should be viewed not merely as a school subject but as a valuable part of out-of-school life.

Planting Bulbs in Miss B's Class

The selling of bulbs in school is a familiar situation which makes mathematics a real, vital experience for children. It affords many opportunities to develop mathematical concepts, to reinforce others, and to develop the ability to meet number situations which arise in connection with classroom activities.

There was a great deal of action in Miss B's class when the time came for the children to place their bulb orders. Here was a class project in which all could participate. The children talked it over, planned together, and came to the decision that it would be best to work in groups.

Sally's group was chosen to handle the money. A specific time was set for this task each day. This was a very important job. The children would have to make change, stack coins in piles, and count the money. Narcissus bulbs sold for 5¢ each, crocus bulbs for 3¢. The children, with Miss B's help, made up a price list to be used as a guide by this group and the class as a whole.

Tim's group was in charge of the order slips. The children had to check every slip to see that it was filled out correctly and that the total amount of each was correct.

Name:	Jane Smith		
Class:	Miss B		
Order	No.	Cost	Total
Narcissus	2	5¢	10¢
Crocus	1	3¢	3¢
Total Amount			13¢

* Misses Nadelman and Paskins are elementary school mathematics coordinators in districts 21 and 22 of New York City. This unit was worked out in Assistant Superintendent Johanna Hopkins' district.

	1	2	3	4	5	6	7	8	9	10
NARCISSUS—5¢ each	5¢	10¢	15¢	20¢	25¢	30¢	35¢	40¢	45¢	50¢
CROCUS —3¢ each	3¢	6¢	9¢	12¢	15¢	18¢	21¢	24¢	27¢	30¢

Bob's group had to be ready to get the bulbs when they arrived. A very large basin would be needed, much larger than the one used to wash the board. Perhaps two would be needed. These children would have to check the bulbs to be sure they received as many as had been ordered.

Emily's group was to distribute the bulbs. They worked very closely with Tim's group, the children who were responsible for checking the order slips.

Sally's and Tim's groups worked hard. They kept very careful accounts which they checked with Miss B every day. All this was completed by the end of the week. The money was changed into bills and sent off to Mr. K's office. The class order sheet went along with it.

tom. Everyone would be able to see just what was happening. The yellow bowl looked large enough for two. It was quite wide. The bulbs would not be too crowded. The children had to use good judgment and give their reasons for the selection of the containers.

Miss B had her pebbles and fiber ready. She put one bulb into a jar about three-fourths full of water; another into a jar about half full of colored pebbles and water; the two for the bowl were planted in fiber.

The children were now ready for the next step. Miss B's bulbs had to be placed in a dark closet for about three weeks.

Class: Miss B			Room 302
Bulb Order			
	No.	Cost of Each	Total
Narcissus	35	5¢	$1.75
Crocus	18	3¢	.54
Total Amount			$2.29

November

Sun.	Mon.	Tues.	Wed.	Thur.	Fri.	Sat.
	1	2	3	4	5	6
7	8	9	10	11	12	13
14	15	16	17	18	19	20
21	22	23	24	25	26	27
28	29	30				

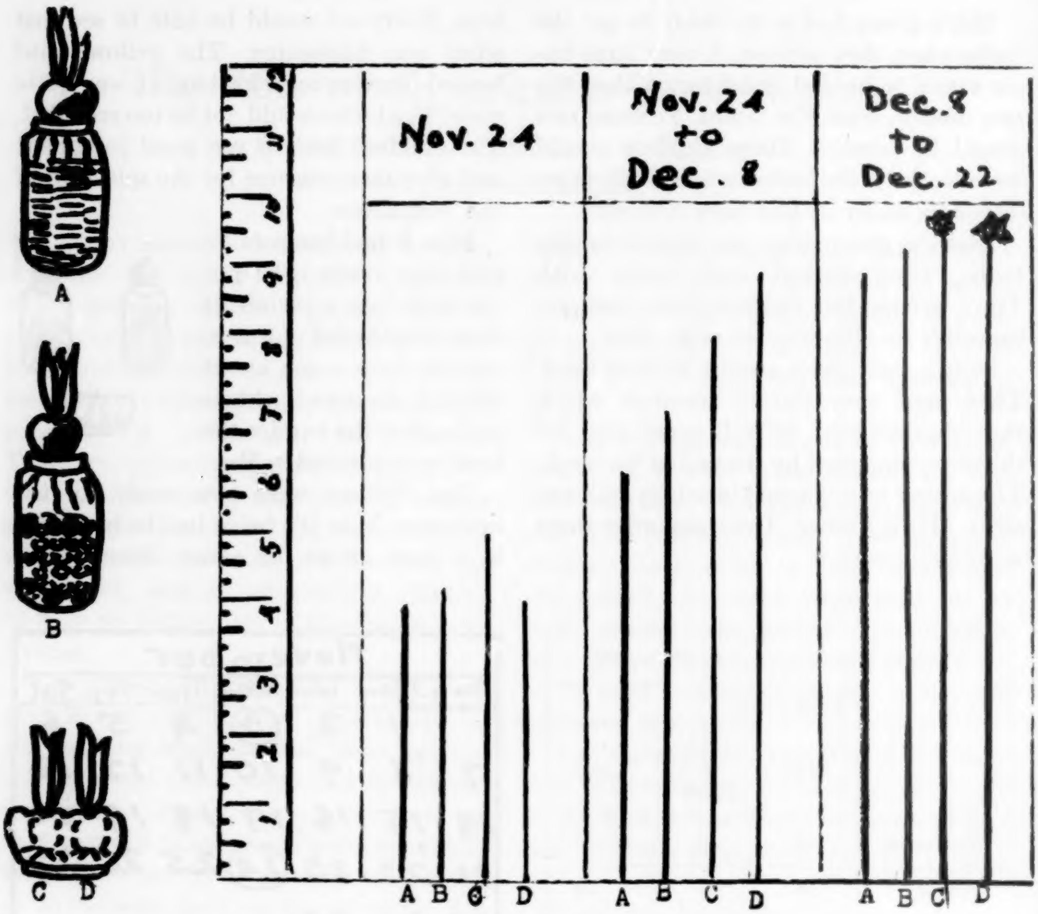
When the long awaited notice came to let the class know the bulbs had arrived, Bob, Emily and their helpers went into action. Each child in the class received exactly what he had ordered—no errors, no tears.

The bulbs were compared as to size and shape. Carol noticed that her narcissus was much larger than Dick's crocus. She wondered if her flowers would be larger; if her plant would grow taller.

Miss B decided to plant her own four bulbs. It would be fun to watch them grow. The children were anxious to help her. Were there any containers in the room that could be used? Those two jars looked just right. The checks were small enough so that the bulbs would not fall to the bot-

They went in on November 3rd. The calendar was used to find out when they would be ready to come out of their hiding place. November 24th was the day.

When Miss B's bulbs were removed from the closet, the children examined them very carefully. They had not all grown exactly alike. Some shoots were taller than others. It was decided to keep a record to show which bulb thrived best; the one in the water, the one in the pebbles, or one of the two in the bowl of fiber. Every two weeks the bulbs were measured with a twelve-inch ruler and the results recorded on a chart. A different color line was used to show each plant's growth. In printing these are shown by letters A, B, C, D.



The two plants which had been planted in the fiber bloomed first. The children decided to share what they had discovered with other classes and explain their graph. Children volunteered to do this.

The activity was one in which arithmetic played an essential part. The children had opportunities to check quantities; to use money, and to learn the purchasing value of the various coins; to stack money to facilitate counting; to make change. Ordering involved careful writing of numbers, tallying, and addition. The children were keenly aware of duration of time and sequence of events. They used the calendar as a means of measuring and recording time. They learned to make a graph and to refer to it to get information.

Editor's Note. In an experience unit like this one dealing with bulbs, a good deal of "thinking and discovery" arithmetic is involved. Also there is ample opportunity for coordinating arithmetic with areas such as language arts, science, and social studies. The big problem in an experience involving a whole class is to get each pupil to think independently so that his learning really belongs to him and is not merely the repetition of the learning of the more able pupils. Experienced teachers know how to follow through an experience unit and provide the needed practice so that the products of thinking and discovery will be retained and applied.

The Abacus as an Arithmetic Teaching Device

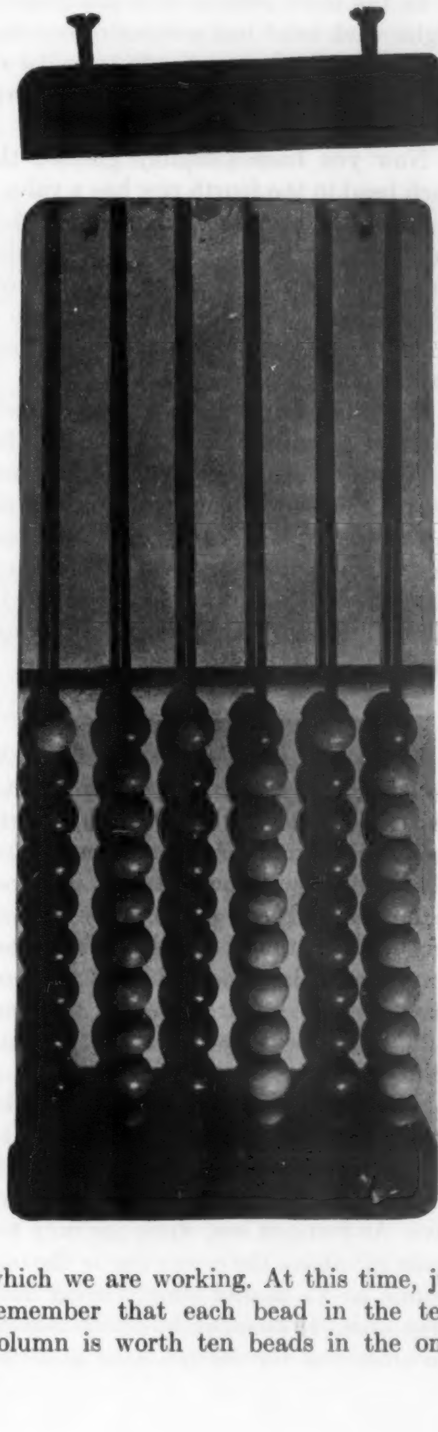
Robert W. Flewelling
Harvey Scott School, Portland, Ore.

The instructions that you are about to read should help you to understand arithmetic better by showing you, step by step, how different kinds of problems can be done on the oldest kind of adding machine in the world. Because each step will be harder than the one before it, you should work each problem with pencil and paper to help you understand as you go. It would also be a good idea to make up some extra problems to check your understanding. If you have trouble anywhere, you will probably find that you have not understood something in an earlier step.

The counting frame (or abacus) is a simple frame holding parallel rows of beads. On our abacus, there are ten beads in each row—nine of one color and one of a different color. If you will hold the abacus so that the rows of beads are up and down with all of the beads pushed up above the center bar and the different colored single beads on the top, you will be ready to use it.

As you look at the abacus now, the row of beads on the extreme right represents the "One's column." This means that each bead in this first row has a value of one. Pulling the beads down to the bottom of the abacus "writes" the numbers. Thus seven beads on the lower half of the first row would stand for seven ones, or as we say in reading it, seven.

Now look at the second row of beads from the right. Each bead in this row has a value of ten of the beads in the one's column. We can compare these beads to coins. We know that ten pennies have the same value as one dime and that a dime is equal to ten pennies. Later we shall learn to use this idea to "trade" ten one's for one ten or one ten for ten one's. This will not change the value of the number with



which we are working. At this time, just remember that each bead in the ten's column is worth ten beads in the one's

column. Therefore, if we pull three beads down to the bottom in this row, we "write" three tens or, as we say, thirty. These three tens then, with the seven ones that we pulled down first, make the number thirty-seven.

In the third column of beads from the right, each bead has a value of one hundred. Pull down two beads to make our written number read two hundred thirty-seven.

Now you have probably guessed that each bead in the fourth row has a value of one thousand, in the fifth row ten thousand, and in the sixth row one hundred thousand. You could go on with millions, ten millions, hundred millions, and so on indefinitely if you had a long enough frame to hold enough strings of beads.

With this simple machine, you can add, subtract, multiply and divide. In fact hundreds of thousands of businessmen around the world are using it this very day in their work. There may even be a business in your neighborhood where one is being used. It might not look just like the one we are using, but it works in the same way.

Addition

Addition is easy with the abacus. We already have two hundred thirty-seven "written." Let's add five hundred forty-one to this number. All that we need to do is to pull down one bead in the one's column, four beads in the ten's column, five beads in the hundred's column, and count the beads in each column to read our answer: seven hundreds, seven tens, and eight ones—or seven hundred seventy-eight.

Now, if you are sure that you understand what we have done so far, let's try adding two hundred eighty-three to the seven hundred seventy-eight that we now have. As you can see, there are only two beads left above the center bar in the ones column so we cannot simply bring down three ones. Remember however that we can trade one ten for ten ones as we explained earlier with pennies and dimes as examples. We can also trade one hundred for ten tens, one thousand for ten hundreds and so on. Notice that the top bead in each column is the same color as the nine beads in the next column to the left. This is to remind us that ten beads in any column have the same value as one bead in the next column to the left. Now, to get back to our problem, if we pull down the two beads that are left above the center bar in the one's column, we can exchange them for one bead in ten's column by pushing them all up again and pulling down one bead in the ten's column. This move added two ones to seven hundred seventy-eight and gave us more one's to pull down; we must still pull down another bead in the one's column to make a total of the three ones that we are adding to our original value.

We need to add eight tens now to our partial sum. Since we have two beads left at the top in the ten's column; pull down the two tens, push all of the tens back up, and pull down one hundred. This added two tens to our partial sum but we needed eight tens so we can now pull six tens back down to make eight tens added.

There are now two beads left above the center bar in the hundreds column. To add two hundreds, pull down the two above the line. Our number now reads ten hundreds, six tens and one one, but ten hundreds are the same as a thousand. Therefore, we push all of the hundreds back up and pull down one thousand. Our number now is one thousand, no hundreds, six tens and one one which we read as one thousand sixty-one.

In writing this number on paper, we would have to put a zero in the hundreds place to show that there are no hundreds in it. This we can call "holding the place" of hundreds for, as you can see on the abacus, each place has a value whether it is occupied or not. If a place is occupied, we call the number that occupies one place a digit. The invention of the zero to show that a place is there, even though it is not

occupied by a significant digit, was one of the most important steps in the development of our modern arithmetic.

Subtraction

Subtraction is just the opposite of addition. Adding is putting one number with another; subtraction is taking one number away from another. Let's start by subtracting two hundred twenty-eight from four hundred forty-eight. Starting with all the beads at the top as in addition, pull down four hundreds, four tens and eight ones. Then push eight ones, two tens and two hundreds back up to the top. This leaves our answer of two hundreds, two tens and no ones which we read as two hundred twenty.

That was an easy problem. If you are sure that you understand how it was done, we shall now subtract eighty-seven from the two hundred twenty that we had left. We can't take away seven beads from the one's column as it is, because there are no beads below the center bar. We can trade one ten for ten ones just as we traded ten ones for one ten in addition without changing the total value of the beads below the line. Therefore, we can push up one ten and pull down the ten ones. Now we can take away seven ones by pushing seven beads back up to the top. There is one bead left in the ten's column but we have to take away eight tens. Take away the one bead in the tens column by pushing it up and then trade one hundred for ten tens by pushing up a hundred and pulling down ten tens. Now, since we have already taken away one ten, push seven more tens back up to the top to make the eight tens that we are subtracting. This leaves our answer at the bottom: one hundred thirty-three.

Now if you understand what we have done so far, clear the abacus and try this one: take nine from one hundred on the abacus. This one is a little tricky. There are no beads "written" in the one's column so we can't take nine away as it is. However, there are no beads in the ten's column either so we can't borrow there. There is one bead in the hundred's column so we can trade it for ten tens by pushing it up and pulling the ten tens down. Now we can trade one of those tens for ten ones by pushing one ten up and pulling ten ones down. We can now take away the nine ones by pushing them back up. Our answer is left at the bottom again, nine tens and a one, or, ninety-one.

Multiplication

We have already seen that subtraction is the opposite of addition. Now we shall see that multiplication is closely related to addition. As a matter of fact, we could say that multiplication is a short form of addition. Let's use a simple example: 4×345

We work this:

$$\begin{array}{r} 345 \\ \underline{\times 4} \\ 1380 \end{array}$$

but we could work it:

$$\begin{array}{rl} 345 & \text{(using 345 as an addend 4 times)} \\ 345 & \\ 345 & \\ \underline{345} & \\ 1380 & \end{array}$$

Similarly, we can do any other multiplication problem by using the multiplicand as an addend as many times as is required by the multiplier. It would, of course, be a long tedious process to do any very difficult problem this way on paper, but this is basically the method used by a modern calculating machine in doing multiplication problems. Let's do another simple example: 234×345 We would do this problem on paper:

$$\begin{array}{rr} & 345 \\ & \underline{234} \\ 4\times345 & 1380 \\ 30\times345 & 10350 \\ 200\times345 & \underline{69000} \\ & 80730 \end{array}$$

On paper, we usually leave off the zero shown at the end of the ten's product, the

two zeros at the end of the hundred's product and so on. We do move those products over the necessary spaces, nevertheless, which accomplishes the same purpose as the space holding zero.

To do this problem the long way we could use 345 as an addend 234 times just as we used it four times in our first example of multiplication. The following solution may suggest a simplified method of getting the same results:

1×345	345
4 times	345
	345
	345
10×345	3450
3 times	3450
	3450
100×345	34500
2 times	34500
	80730

As can be seen, we can work this problem in this way on the abacus by first adding one times the multiplicand the number of times indicated by the digit in the one's column in the multiplier. Next we add ten times the multiplicand the number of times indicated by the digit in the ten's place in the multiplier (in this case we do this by using 345 as an addend three times on the tens, hundreds, and thousands column and carrying to the ten thousands column when it becomes necessary). Last we add one hundred times the multiplicand the number of times indicated by the digit in the hundred's place in the multiplier (in this case we use 345 as an addend two times on the hundred's, thousand's, and ten thousand's columns.

This process takes more time than it would to do the same problem on paper, but a modern calculating machine is comparatively slow in working this type of problem also. The reason is that a calculator does the problem just as we do it on the abacus.

Division

Division like multiplication is a short-cut. Just as multiplication is a short way to do addition, division is a short way to do subtraction.

When we say, "Divide 147 by 12," we are saying in effect, "How many 12's are there in 147?" We could find the answer by subtracting 12 from 147 as many times as possible and counting the number of times we subtracted (which would give us our quotient) and any number smaller than our divisor that was left over would be the remainder. This would be doing our problem the hard way.

A simpler method of doing it should suggest itself in the following solutions:
On paper we would divide:

	12
	12)147
10×divisor	−120
	27
2×divisor	−24
	3

But we could subtract

	147
10×divisor	−120
	27
1×divisor	−12
	15
1×divisor	−12
	3

As you can see, we could subtract one or ten or one hundred or one thousand or ten thousand, etc. times the divisor as many times as possible using the largest of such values that can be subtracted from our dividend as our first subtrahend (the number of times we subtract at this stage is the first digit in our quotient). Then we can subtract each smaller multiple of ten in turn as many times as possible to find the subsequent digits in the quotient. When our remainder is smaller than our divisor, if there is a remainder, that value is the remainder for our problem.

We can now do our division problem in this way on the abacus and from this example we can generalize the method for doing any division problem.

To divide on the abacus, first "write"

the dividend below the center bar by pulling down the required number of beads in each column. Then, *starting from the left*, use only as many columns of beads as you must in order to give a number larger than the divisor (we might call this number a partial dividend). Subtract the divisor from this partial dividend until the remainder is smaller than the divisor or zero. The first digit in the quotient is the number of times you subtracted the divisor from this first partial dividend. Then move to the right and include as many more columns as you must in order to provide another partial dividend larger than the divisor. If more than one extra column is needed, insert a zero in the quotient for each additional column (other than the first). Once again subtract the divisor from this partial dividend until the remainder is smaller than the divisor. The number of subtractions from this partial dividend is the next digit in the quotient.

The foregoing processes are repeated until the dividend has been "used up." The final remainder will then be the remaining beads below the center bar (which must, of course, represent a number smaller than the divisor). The quotient will be the accumulated series of digits representing the number of subtractions with the first partial dividend and with each added column moving to the right.

EDITOR'S NOTE. Mr. Flewelling has written the directions for a demonstration so clearly that any teacher should be able to follow them. There is a certain amount of drama and romance in learning to use a device such as an abacus. Many Orientals who are not familiar with the Hindu-Arabic number system are more proficient with an abacus than most of us are with our numerals and with our best calculating machines. However, the greatest value in using an abacus comes from gaining insight into the number system and our modes of calculating. Frequently the modes of "figuring" which pupils themselves develop are very similar to those employed several hundred years ago before modern algorisms became so firmly established.

The illustration shows how Mr. Flewelling's abacus was made. He has inserted six wires into end blocks and screwed these to a rigid backboard. Perhaps large colored beads can be purchased in your locality. The other materials are easily obtained and assembled. Do you like his suggestion that the tenth bead in each column be the same color as the bottom nine beads on the column to the left? We must always remember that any device such as an abacus is worthwhile only if it is employed so that desirable learnings ensue. Why not try Mr. Flewelling's demonstration with a class, at a teachers' meeting, or at a parents' meeting.

Who Counts?

WILLIAM S. HICKEY
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ONCE UPON A TIME there was a little, three-year-old, yellow-haired boy, whose mother and father were sad because the little boy didn't know how to count.

"Look," they said. "Listen to him!"

"Count for the man," they said. "Go ahead, one—two—three— . . ."

And the little boy smiled, and said, "One—three—fourteen—six—twelve."

The two parents looked at each other. They shrugged their shoulders. They raised their eyebrows. "See," they said. "He doesn't know how to count. It must be hereditary," "He's so young, too, poor little kid."

So they went along hoping that maybe the little, yellow-haired boy would be talented in art, maybe, or be able to speak four languages, because they were sure he'd never be able to do arithmetic.

But the little, yellow-haired boy, and his puppy dog that went to bed with him, smiled and were happy because they knew. They knew very well that the little boy COULD count.

The little boy pointed at his blocks and counted, "One—three—fourteen—six—twelve. . . ."

But the mother and father looked at each other. They shrugged their shoulders.

"It must be hereditary," they said.

So they went along hoping that maybe the little, yellow-haired, three-year-old boy would be talented in music, maybe, or be able to whistle through his teeth, because they were sure he'd never be able to do arithmetic.

But the little, yellow-haired boy and his puppy dog that went to bed with him, smiled and were happy because they knew. They knew very well that the little boy COULD count.

The little boy pointed at his toes and counted, "One—three—fourteen—six—twelve. . . ."

The parents looked at each other and shrugged. Then they looked at the little, seven-year-old girl from next door. They said, "Listen to him."

"He doesn't know how to count," they said. And they shook their heads.

But the little-seven-year-old girl from next door was thinking of something else. She said, "May we have some cookies? I like cookies."

And the little, seven-year-old girl climbed on a chair and lifted down the big cookie jar. "Now, We'll have some cookies," she said.

Since she was a polite little seven-year-old girl, she offered the cookie jar to the little, yellow-haired boy and said, "Have some."

The little yellow-haired boy smiled and took three cookies, because three cookies fit so nicely in his hand.

"Look," said the little, seven-year-old girl, "he took three cookies."

The parents of the little, yellow-haired boy looked at each other. They nodded their heads. "SHE can count." they said. "Why can't our boy? It must be hereditary," they said.

The cookie jar was so big that the little, seven-year-old girl could not put her hand in the jar and hold it, too. So the little, seven-year-old girl said to the little, yellcw-haired boy, "Take some cookies out for me. Give me just as many as you have, "she said.

And the little, yellow-haired boy smiled, and he reached into the cookie jar and took out a cookie. Then he reached in again and took out another cookie. Then he reached in again and took out another cookie. And the little, yellow-haired boy gave the three cookies to the little seven-year-old girl.

The parents looked at each other. They raised their eyebrows. They said to the little, yellow-haired boy, "Count your cookies."

The little yellow-haired boy looked at his three cookies and said, "One—three—fourteen."

EDITOR'S NOTE. Many youngsters who learned rote counting from their parents or playmates and say the counts in correct sequence have no better understanding of counting and its significance than the little yellow-haired boy. To him "three—fourteen—six" might be entirely satisfactory in sequence and he might even have a good "one-to-one correspondence" concept.

Take the Folly Out of Fractions

Joseph J. Latino
Northport, New York

One of the obvious tenets of modern arithmetic teaching is that we must not encourage children to repeat rote style a manipulative process without meaning and understanding. At least the pupil ought to be able to explain in a reasonable manner whatever processes he performs with numbers. While there are other facets in the meaning theory of arithmetic instruction, many people are spending time attempting to lend at least some reason to the mechanical processes every child seems to learn.

As evidence of this witness Edwin Eagle's[1] caution in the October, 1954 issue of this magazine, "Don't Let That Inverted Divisor Become Mysterious," followed by Sam Duker's[2] "Rationalizing Division of Fractions" in the December, 1954 issue. Both articles show concern for the ridiculous practice of teaching children that in the problem 3/4 divided by 2/5 we are only fooling and we meant multiply. What is more, can 2/5 be changed to 5/2 with impunity? Mr. Duker suggests that we teach children the simple expedient of multiplying both fractions involved in the division of fractions by the same number, specifically the reciprocal of the divisor. There is no "flipping over" of numbers (as if it does not matter that 2/5 is not worth 5/2) and no distortion of the injunction to divide.

The concern for the division of fractions is symptomatic of the desire to clarify many of the things we continue to teach children concerning fractions. Therefore, we might address ourselves to the larger problem of methods of teaching fractions in general. This article will attempt to:

(1) Examine some of the words and ideas we presently use in the teaching of fractions.
(2) Discuss a method of defining the fraction for the child.
(3) Discuss a method of beginning the addition and subtraction of fractions.

Verbiage in the Teaching of Fractions

In arithmetic many of the words we use confuse understanding. Let us take the term "least common denominator." It is questionable whether this label makes for sound arithmetic. For example, consider the addition of the fractions 2/3 and 3/4. Most children would say that the "least common denominator" is 12, which is, of course, correct. But have we taught them anything meaningful that will help them understand what they are doing. Indeed, have we taught them something that makes sense? This observer thinks not.

First of all, if denominator means *name of the part* or *size of the part of the whole*, we agree that common denominator implies the *same* size or kind of part. It is doubtful whether this is ever clear to children. But let us assume that it is. To most everyone "least" means smallest. Then is the child right in saying that this is the smallest kind of part or the smallest size into which we can break 2/3 and 3/4? Hardly, since 24ths are a smaller size than 12ths. In fact, for every "least size" which the reader can name, we can name a size that is "more least." It is clear then that 12ths is not the least from the standpoint of size, which is the meaning of denominator. It is equally clear that it is not the least common of the denominators we might choose. No de-

[1] Eagle, Edwin, "Don't Let that Inverted Divisor Become Mysterious," The Arithmetic Teacher, October, 1954.

[2] Duker, Sam, "Rationalizing Division of Fractions," The Arithmetic Teacher, December, 1954.

nominator is any less common than any other as used in this context.

The only valid interpretation of the phrase "least common denominator" is to insist that it is the smallest "number" on the bottom into which both thirds and fourths may be conveniently broken. It is true that 12 is a smaller number than 24. But, when we are adding fractions, denominators are not numbers in the ordinary sense. If they were, we could add them as ordinary numbers. Thus:

$$\frac{3}{4}+\frac{3}{4}=\frac{6}{8}$$

The very reason we cannot add the denominators in this fashion is that they are not numbers but names of sizes. We cannot consider 12 a number in one breath and not a number in the next breath and make sense to children.

It might be noted here that in later mathematics the cause of much of the difficulty in the study of variation can be traced to this encouragement given the idea that 12 as a denominator is smaller than 24. Variation, especially inverse variation, is largely dependent on the notion that as 3/4 changes to 3/8 the whole fraction becomes smaller. While it is true that we say "as the denominator increases the value of the fraction decreases," we are usually considering quantities such as current in the relationship $R=V/I$. In the elementary school we are dealing with a relatively unsophisticated mind. Consistency helps the early learner. If he learns that thirds are larger than 12ths in the concrete stage, his methods of manipulation must reflect that truth, not deny it.

"Least common denominator" encourages another questionable able reaction in children. Place the fractions 1/3 plus 3/4 in front of a group of ninth graders. Ask them if we can make the "common denominator" seven. Invariably the answer is no. Indeed the insistence that it must be twelve is deafening. Yet, to the mathematician there is no law prohibiting us from finding out how many sevenths there are in 1/3 and 3/4. To the adult mind it is obvious that we prefer to change this problem to 12ths because in that form it is much easier to handle. If we changed the fractions to sevenths, we would have fractional numerators. But that does not mean we cannot change any fraction to whatever type of parts we wish. Sueltz in his test series on "Functional Evaluation in Mathematics"[3] has a problem involving the fraction $3\frac{1}{2}/7$. It is interesting to note the concern with which ninth graders regard the problem.

Not being capable of mature judgments, children should be carefully shown that we choose 12ths because we want a convenient part. That fifths or sevenths are also possible must be made clear. This would seem like good, sound mathematics.

Or consider the phrase "reduce to lowest terms." What does this mean to a child, or more important, can it mean anything at all, even to an adult? We shall use the fraction 6/8 as a case in point. If we give the direction "reduce to lowest terms," everyone says 3/4. At this juncture, ask children if 3/4 is smaller than 6/8 and not a few say yes. After all they have "reduced" it. Usually most children see that this doesn't make sense once they have thought about it even for a minute. Well, then what have we reduced if the fraction is still worth the same. We certainly haven't reduced the size of the parts. Fourths are larger than eighths. We have reduced the number of pieces which is probably what "lowest terms" was meant to imply. Yet, is there a reader bold enough to assert that this is the meaning that children attach to the phrase? If there is, his experience has been radically different than the writer's.

Finally, we might cite an example of inconsistency in the teaching of the addition of fractions. In the problem 1/2 plus 3/4 we have the child think about unlike

[3] Sueltz, Ben A., "Functional Evaluation in Mathematics," Test I—Quantitative Understanding, Educational Test Bureau, Educational Publishers, Inc.

parts first. He then changes one fraction to an equivalent fraction with like parts, 2/4. He now has 5/4. Then he divides 4 into 5 and gets 1 1/4. This last maneuver requires an entirely different interpretation of the fraction than the interpretation which requires him to change to like parts. The child is now using the second meaning of the fraction, an expressed division. (Buckingham[4] gives an excellent account of the various interpretations of the fraction.) While this is good mathematics, the good teacher should be aware of this switch in interpretation and take great pains to point it out. But first there certainly should be some work of the type: 5/4 is 4/4 and 1/4. 4/4 is 1 therefore 5/4 is 1 and 1/4. This is consistent with the meaning of fourth. To then demonstrate that we can achieve the same result by regarding 5/4 as a division problem, would be, it seems, good teaching.

The Anatomy of a Fraction

It is surprising to find many ninth grade children who cannot explain what the top number of a fraction represents or what the bottom number represents. They can easily repeat the words numerator and denominator and believe that they have said something meaningful. But when pressed as to the significance of these labels in terms of understanding what they are doing, nearly every ninth grader that the writer has interviewed is hard pressed for an adequate explanation. Numerator tells how many and denominator tells what kind or type. The Latin origins of both words are obvious. It seems that children ought to know at least the meaning of words they use very freely.

Of what does a fraction consist? Early in the school experience of a child he ought to have in his hands an object such as a piece of fiberboard two feet long which represents unity or "one" or a whole thing.

1

Accompanying this basic piece would be two pieces one foot long, each labeled one half.

$\frac{1}{2}$	$\frac{1}{2}$

Then four pieces each labeled one fourth.

$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

In the beginning the teacher might take the option of blank pieces with no labels. Beaverboard can take chalk writing and the labels can be applied as the class is able to identify the fractions by comparison to the original basic unit. For example, this piece is one half the big piece. When assembled side by side on an easel or on the child's desk these pieces would appear as shown.

<table>
<tr><td colspan="4">1</td></tr>
<tr><td colspan="2">½</td><td colspan="2">½</td></tr>
<tr><td>¼</td><td>¼</td><td>¼</td><td>¼</td></tr>
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In the discussions that would certainly follow there are obvious teaching situations concerning the relationship between these pieces. But the teacher must channel the discussion to the meaning of the written symbol of a fraction, say 3/4. Certainly most children will have seen this symbol many times both in and out of school before this projected discussion would occur. They are then ready to look into the make-up of the fraction.

Any child can be asked to write on the board what the label or name of a particular piece is. Let us take the fiberboard representing one-half. The child might write the symbol 1/2 or he might write

[4] Buckingham, Burdette R. "Elementary Arithmetic, Its Meaning and Practice," Ginn and Company, 1953, pp. 242–246.

"one half" or he might even write "1 half." He might say "one half" and not be able to write it in either arithmetical symbols or words. In any event expert questioning by the teacher would bring out the following points:

(1) When we write "one half" we are saying we have one of this particular size (in relation to the whole).
(2) When we write $\frac{1}{2}$, even though we write the word *one* as "1," we are using 2 as a shortcut for saying a half. (The mathematical meaning of one divided by two would be confusing at this point. We are attempting to bring meaning to the reading of the fraction as it is usually read, that is, one-half.)

The children ought to discuss why it is important to record correctly the number of the things we have. Holding up two of the fourths and then three of the fourths makes it clear that two of anything and three of anything (even in fractional parts) is not the same thing. Then the children ought to discuss why it is important to record correctly the size or type of part we have. By comparing one piece representing the half and one piece representing the fourth, most children can see the necessity of being sure that just saying "I have one piece" is an inadequate description. The size of the piece is important.

Several periods of this type of discussion should lead to the generalization that the top number of a fraction represents how many parts we have and the bottom number represents what kind of parts we have. When we write 3/4 , we are saying we have three of this kind or size. Considerable drill might follow in answering such questions as:

(1) How many pieces do we have in this fraction?
(2) What kind or size pieces do we have in this fraction?

The writer thinks it is abundantly clear that this method of presentation is superior to the method of explaining where the numerator and denominator are and then plunging into "least common denominators" and other specialized terminology.

We have noted previously that a fraction is also an expressed division. This meaning can be explored more easily. We can ask anyone to divide six by three by writing 6/3 or we can ask anyone to divide three by four by writing 3/4. That the bar means division is easily accepted much as + means add. These symbols do not have the organic importance to later processes with fractions such as addition that the correct meaning of numerator and denominator have.

Addition and Subtractions of Fractions

Having at least explored the message in the symbols 3/4 and 1/2, a child is ready to look into the matter of combining or adding them together. By this stage our fractional board would contain eighths and possibly sixteenths.

The question now is, "What do we get when we add 1/2 and 1/4?" Most of the children readily see from the board that we get 3/4. But there are certain subtilties in this quick conclusion which form the foundation for understanding the whole process of addition and subtraction of fractions. For example, we have one of one thing and one of another. (Teacher would hold up the necessary half and fourth.) Is it not true that we have two pieces? How can we say we have three pieces which is what 3/4 means? Can we say we have two halves or two fourths? No, there is one half in one hand and one fourth in the other. Why can't we just add and say one and one are two? We need not labor the point but certainly most children will agree that to add these things we must have the same kind of parts. Thus, to have 3 pieces of the fourth size and 3 more pieces of the fourth size on hand and to have the children record the total number of 6/4 reinforces the idea. As we get into more advanced fractions such as 2/3 plus 3/4, examples on the child's level might also include the problem of adding two boys and three girls. We now

have five of what thing? Certainly not boys, nor girls. Each label or name would be incorrect. We have to find a new label to represent the total, five people.

That this fundamental meaning is not clear to children who have had eight years of arithmetic is well known to the ninth year teacher who asks, "In the problem 3/4 plus 3/4 why can't we add the bottom numbers?" But can we blame the children when probably their first brush with addition of fractions in a formal way involved "common denominators" and a process one student referred to as the "backwards Z" method? This type of mumbo-jumbo is hardly conducive to meaningful addition of fractions.

Breaking Fractions up into Smaller Pieces

If the child realizes that he cannot add 3/4 and 1/2 in their present state because they are not the same size part, he is immediately confronted with the problem of changing them into identically sized pieces. His experience almost intuitively tells him 1/2 is two fourths. And we ought to proceed with this type of situation for a while in order to give the pupil time to accustom himself to the necessity of breaking some fractions into smaller pieces. The fraction board is of great value here. He can literally see that an eighth is smaller than a fourth. In due time he arrives at the important generalization that to break a fraction up into other pieces for addition purposes usually involves breaking it up into a larger number of smaller pieces. We cannot hurry understanding at this stage. We must be reasonably certain that every child sees that we can consider changing a fraction to other size pieces, and, more important, we must be certain that every child actually does this with fiberboard pieces.

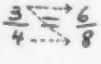
But somewhere a method has to be developed that will enable the child to change a fraction such as 2/3 to 12ths without referring to a fractional board. We can stimulate this predicament by having him try to add 1/2 and 1/3 without any fiberboard pieces representing sixths. Questions now arise:

(1) What size pieces shall we break these into?
(2) How many will we get of each from $\frac{1}{2}$ and $\frac{1}{3}$?
(3) Can we add them the way they are? (Review of a concept)

Again without laboring the point the teacher must lead to the manipulative axiom fundamental to all work with fractions. *We may always multiply top and bottom of a fraction by the same number and be sure the resulting fraction has the same value as the fraction with which we started.* This will take time but it is worth it. Thus, the child who knows that 1/2 equals 2/4 can be made to see that if he multiplies top and bottom of 1/2 by 2 he achieves 2/4 which he knows is the same value.

It may be argued that teaching this manipulation introduces a break in meaning and an artificial device. Yet, it is a sound mathematical maneuver used extensively throughout all of mathematics. The child does have the assurance that it does produce reasonable results because he sees on the fraction board that 1/2 is 2/4 or that 3/4 is 6/8. He can have reasonable faith in its use. Indeed some might even note from the fraction board the reason for its effectiveness. But this can be left for a more mature age.

Certainly this method is superior to the method earlier referred to as the "backwards Z" method. While most of us do not use this colorful name, we were probably taught to change fractions in this manner. Take the fraction 3/4 which we wish to change to one with the denominator 8. We divide 4 into 8 and get 2. We then multiply the three by 2 and get 6 which we put over the eight. If we trace the path of operations we can note the origin of the term "backwards Z." Some may contend that this is no more artificial than multiplying top and bottom by the same number. Perhaps, but certainly the "Z" method does not have the sound uni-



versal application to fraction work that the latter does. Indeed, with the tool of multiplying or dividing top and bottom of a fraction by the same number, every process with fractions can be accomplished. It is the one thing we can do and be sure we have retained the original value of the quantity we set about to break down. Buckingham calls this law the "Golden Rule of Fractions."[5] Its universal applicability to fraction work makes it a valuable idea to give children for all their future mathematics.

Summary

In 1928, a long time ago, Brueckner made an exhaustive analysis of error in fractions compiling tremendous tables itemizing each type of error.[6] The total number of errors analysed was 21,005. Such items as lack of comprehension of process involved, difficulty in reducing fractions to lowest terms, difficulty with improper fractions and computational errors were investigated. Brueckner observes, "All pupils make errors for a great variety of reasons. The reasons vary from lack of knowledge of fundamental facts to peculiar methods of work due to such psychological difficulties as faulty attention span, roundabout procedure and lack of power."[7] In the intervening years we haven't done too much about the peculiar methods of work, roundabout procedures and lack of knowledge in the realm of fractions. Our processes with fractions remain about the same.

This paper, which restricted itself to certain basic facts up to the multiplication of fractions, is a plea to:

(1) Eliminate many of the terms and ideas in fraction work which tend to confuse children. Among these are the use of the words numerator and denominator at too early an age, the phrases "least common denominator," "reduce to lowest terms" and "invert."

(2) See that before a child deals with fractions he understands two interpretations of a fraction (a) so many of a particular kind, and (b) divide the top number by the bottom number.

(3) See that before a child adds or subtracts fractions he must understand that we can only add like things. The bottom number represents sizes of parts and cannot be added in the ordinary sense. We may change so many of any particular kind to any other kind we wish. We choose particular kinds because they are easier to work with.

(4) Give a child a concrete aid such as a fraction board so that he may visualize the meaning of the numerator and denominator before he deals with fractions.

(5) Emphasize the importance of the fractional axiom, that we may multiply or divide the numerator and denominator by the same number without changing the value of the fraction.

(6) Proceed more slowly in the earlier stages of fraction work. Be certain of a background of handling fractional material that will provide the basis for later manipulations with fractions. Use these aids so that they give meaning to computational methods.

[5] *Op. cit.*, p. 249.

[6] Brueckner, Leo J., "Analysis of Errors in Fractions," *Elementary School Journal*, 28: 760–70, 1928.

[7] *Ibid.*, page 760.

EDITOR'S NOTE. Mr. Latino points out the need for clarifying the meaning and significance we attach to certain expressions we use in work with fractions. He is concerned with the logical inferences that children are apt to make about such things as "the least common denominator." Is he overly concerned? Do children in the elementary school become befuddled by the expressions which they only partially understand? Should complete understanding be stressed in the elementary school or should the more refined understandings be left for the grades of the junior and senior high school? Do we need some new terminology in fractions? Certainly, Mr. Latino is right in his desire for children to understand and to see sense in what they are doing rather than in merely following rules that adults have established. While children's methods are often cumbersome, these same methods and explanations may often be much more significant to them than the refined techniques commonly used by adults.

An Experiment with Hand-Tally Counters

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WHO EVER HEARD OF GIVING twenty-seven children in first grade their very own calculators with which to work and to learn? It was done, and resulted in a basic concept in place value, a beginning knowledge in the use of such accurate machines, a way to see the processes of various number problems, *and* we played at numbers with the excitement and pleasure and downright good time of a rousing game of Red Rover!

These small machines—we called them "clickers"—are palm-sized devices used in more adult circles for counting such things as numbers of customers entering a store, or attendance at concerts. With each press of the thumb a number appears in one of the four places up to 9,999. They click softly and pleasantly with each application of pressure. The numbers are large enough to be seen by six-year-old eyes easily. They are printed numbers, not manuscript which proved to be worthwhile.

Did you ever see a youngster watching as a grown-up tap-tapped away at a typewriter? What a fascinating, mysterious instrument it seemed—and when the adult had left it idle, the child with the same businesslike air tried the tapping with the same degree of studied efficiency—loving the "grownupness" of it, though scarcely understanding the use.

So it was with the tally counters, but there was a distinct advantage. All the fascination and delight was there, but these small devices were simple enough so that we knew what we were doing.

As a learning device, it has a well-rounded personality. As the eyes see the number, the ears hear the click and the fingers do the pushing to achieve it all. To get each number, then, requires—as they say in education, audio, visual, and kinesthetic responses. The little machine is well-suited to the physical needs of the little ones. It fits nicely over the two middle fingers, and is easy to push.

Naturally, it precludes a well-founded knowledge of what numbers are—something gained only through concrete experiences. But after this foundation is achieved the "clickers" have many and varied uses. This is how they were used in one first grade.

The initial presentation of the device was made with three main points in mind. 1. Gaining a certain facility with the mechanics and use of the machine. For example, learning what "clear your machine" means. 2. Introducing ideas of what the machine was used for in its more usual environment. 3. Getting some of the at-random fascination for just clicking out of our system, so that at the next lesson we could move ahead more rapidly. The results: an excited, happy bunch of busily clicking children. It sounded like an office full of secretaries who loved their jobs and who let oh's and ah's come out with surprising frequency.

The next lesson was the beginning of the teaching of place value—the ten's system. Four little boxes were drawn on the board (roofs and chimneys added just for fun). Above the boxes, or houses as we called them, was written: one's, ten's, hundred's, and thousand's. "Only one's can live in the one's house, and when there are nine one's —well, there's just not enough room for another one—and that family, now *a* ten moves to the next house, the ten's house." We stuck to the one's and ten's houses for the next few lessons. The children pointed out the houses on their individual machines. We began clicking in unison (it

sounded now somewhat like a regimented tap dance school). As we clicked to five I told them the simple story of five. (The story of five is that there are five ones in the one's house.) The excitement was at high pitch when we came to ten, because the one's family just moved out bag and baggage, leaving nothing at all (0), and sure enough, there was one ten family in the ten's house. So it went in future lessons, telling stories of numbers, until the young ones were able to read any number at all starting with the thousand's house and reading down the street, so to speak. It was exciting and thrilling, and they could watch those numbers moving around —for they were right there in the hand, to feel and hear and see.

After this number story telling had been practiced, it seemed there was a haziness about "quantity" of one's and ten's and hundred's. Just exactly what made one *ten* different from one *one* other than the fact they were in different places? This is what we did in first grade to make the difference more clear and to show the quantity of a ten as opposed to the "oneness" of a one. I built a giant machine of sorts in the front of the room. It was simply three bowls of graduated sizes and different colors (pyrex mixing bowls). The smallest was for the one's, (just as the smallest wheel inside the machine is for the one's), the middle-sized was for the ten's, and the largest for the hundred's. As the children clicked, I dropped lima beans into the one's bowl. When there were ten lima beans, they were put into a small paper cup and transferred to the ten's bowl. Here was 1 ten, but 10 ones. In further lessons we carried this to ten paper cups—one hundred lima beans, all in one paper sack, which was put into the hundred's bowl. This procedure was clicked through and the number story was told at each fifth place. "What's the story of 95?" "Well, there are 9 ten's in the ten's house, and 5 one's in the one's house." "How many ones, or lima beans, make 9 tens?" "90."

These are only two examples of the gaining of an advanced concept. There were many uses for these "clickers" in the first grade. Practice with the mechanics of counting. Practice in counting by two's, by five's, and by ten's—always while watching the process even though you skip for speed, the saying of what goes in between.

On occasions when there was a need to count money brought in by the group, the "clickers" helped us know how much money we should have to check against the actual answer we came up with in the counting.

The "clickers" were a boon to the slow learners who considered it a great privilege to go to the back of the room and count in the store. It eased the strain of writing numbers down for them, yet gave the same tie-up between the symbol and the sound. And they were in actuality counting something partially concrete when they counted each "push."

In my opinion, the "clickers" are an expedient device. The results they brought about were worth while and their efficiency in bringing them about was remarkable. They were an incentive as well as an aid to learning. They have an individual appeal and a personal quality. They are expedient. Their relatively high initial cost is about the same as the investment in a good textbook. However, one set of "clickers" could be used by many classes and their life expectancy would be much longer than a workbook or other individual aid to learning.

EDITOR'S NOTE. Yes, the little "clickers" combine audio, visual, and kinesthetic avenues of learning. For certain children the sense of touch seems to quicken the mind. Perhaps it is merely the satisfyingness of touching something but it may be much more than that. Our psychologists have not sufficiently investigated the various modes or avenues of learning singly and in combination. It is easily apparent that pupils working with Miss Hooper enjoy their work in arithmetic. Some teachers may not agree with her early introduction of numbers in the hundreds. However, this seems a natural extension when tally-counters are used and the ten's principle of the number system is being learned.

Requiring Proof of Understanding

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THE AMERICAN SCHOOL is often criticized for failing to help children develop study habits which enable them to do independent, logical thinking. Teachers of a given grade are inclined to blame instructors of the preceding grade for the child's inability to grapple with simple principles of logic and independence of thought. Complaints concerning the child's lack of ability to solve problems dealing with quantitative situations come from many quarters. Often teachers make the remark: "My children do fairly well with the arithmetical computation involved in solving examples but they have much difficulty with the solution of 'verbal problems' in arithmetic." The expression "verbal problems" is generally employed to refer to the word description of a quantitative stituation for which the solution is not indicated. It is to be regretted that arithmetical verbal problems are not solved with less difficulty; however, it is equally regrettable that children do not understand what is involved in the various processes and operations indicated by use of abstract symbols. The fact is that many teachers are not aware of what is actually involved in the several arithmetical operations and processes they attempt to teach. Several practices which may help children to understand better why they do certain things in arithmetic, as well as help them to see what is involved in the fundamental operations performed with the abstract symbols, will be indicated.

Demonstration Versus Discovery

The "demonstration method" of teaching arithmetic as it is practiced in most of the elementary schools today fails in many instances to help the child to see the reasons for the various operations in arithmetic and to appreciate different arithmetical relationships. Performing correctly the steps required for solving problems and examples does not always indicate complete understanding on the part of the child. One should, however, appreciate the fact that the "demonstration method" develops, at least, some understanding of arithmetic.

The "discovery approach" to the teaching of arithmetic, which is advocated by many writers in the field, places much emphasis upon meaning and understanding. Meaning is generally defined as seeing reasons for what is done, while understanding is defined as seeing relationships that exist. Perhaps these definitions may be restricted too much to please some persons.

Children should see reasons for what they do in arithmetic and, also, be aware of relationships existing between various arithmetical processes and operations. For example, the child should see it is better under certain circumstances to regroup or rearrange 6 and 7 to become 13 rather than to retain the two distinct groups of 6 and 7. In other words, the child while learning the addition facts should see that addition in the arithmetic program is a process of regrouping or rearranging.

Similarly, it is desirable for children to see existing relationships between certain operations such as between addition and multiplication. No parent or teacher should be discouraged or alarmed about the child who, when he first comes face to face with a quantitative situation that best would be solved by multiplication, resorts to addition to arrive at the correct answer. Any understanding person should readily see that the answer to 6×7 can be obtained by $7+7+7+7+7+7$. Since the

child has already had the addition facts before encountering the multiplication facts, it is reasonable to suppose that he could arrive at the answers for all the multiplication facts by addition, even though it be a longer process for him. It is not advocated that children should solve multiplication examples and problems by addition; however, teachers and parents should appreciate the ability of the child who sees that 6×7 means six groups of 7's are to be regrouped or rearranged to become 42, which, in terms of the number system, is four 10's and two ones.

In the initial study of the multiplication facts pupils may very well come to understand the process of multiplication better if they see the relationship between multiplication and addition. No doubt one of the chief purposes of the arithmetic program should be to help the child see all these existing relationships. The typical demonstration method of teaching arithmetic in the schools today often enters upon meaningless drill and practice on processes or operations before there is any understanding of what is actually taking place in the processes or operations.

"Proof" of Understanding

How best can the child's understanding of what he does in arithmetic be developed and checked? The advocates of the "discovery method" suggest that the child be required to give "proof" that he understands what the problem or example is asking, as well as to give "proof" that his answer to a problem or example is correct. This "proof" of understanding of an arithmetical process or operation is deemed necessary not only during the introductory stages of learning but also during practice or drill work.

The word "proof" as used in this discussion may be subject to question by various persons; however, for lack of a better term, "proof" in arithmetic work is employed to mean:

1. Simply the rearrangement of objects or marks to show a fact. (Examples are from a child's work.)

Example:

$8+6=14$

$11111111+111111=(1111111111)1111$

$10+4=14$

In this example the child regroups 8 and 6 in terms of the base of the number system. The new groups are one *ten* and four *ones* which is 14.

Example:

$$6\times 7=42$$

1111111 1111111 1111111 1111111 1111111 1111111 = 42

10 + 10 + 10 + 10 + 2 = 42

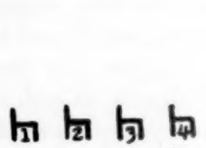
The child indicates in the above illustration his understanding that in multiplication of whole numbers a regrouping is required. He also shows an understanding of the relationship between addition and multiplication of whole numbers. In the above example, the pupil has shown that he knows six groups of 7 *ones* are regrouped in terms of *tens*, the base of the number system. After the regrouping takes place, there are 4 *tens* and 2 *ones*, or forty-two.

2. The use of pictures or other processes to prove understanding of problem.

Problem: John places 4 chairs in each of 3 rows. How many chairs does he use?

Illustration:

$$\begin{array}{r} 4 \\ \times 3 \\ \hline 12 \end{array} \qquad \text{or} \qquad \begin{array}{r} 4 \\ +4 \\ \hline 8 \\ +4 \\ \hline 12 \end{array}$$

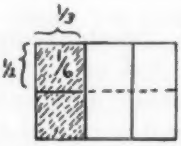


The student shows that he understands what is actually stated in the problem. The teacher can readily see that the child has not merely memorized the multiplication fact of $3\times 4=12$.

3. The use of diagrams to indicate understanding.

 Example:

 $\frac{1}{2}\times\frac{1}{3}=\frac{1}{6}$



In the diagram it can easily be seen that the shaded part is $\frac{1}{3}$ of the whole rectangle. One half of the shaded part is $\frac{1}{6}$ of the whole rectangle. Thus it can be said that $\frac{1}{2}\times\frac{1}{3}$ or $\frac{1}{2}$ of $\frac{1}{3}$ of a whole is $\frac{1}{6}$ of the whole.

Several advantages should result from requiring children to prove their work and understanding by methods similar to the ones presented in the above examples. The following five appear to be the most obvious and significant advantages: (1) The children are not rushed through mathematical processes so hastily that they do not have time to get a real understanding of the processes. (2) The children use concrete and semi-concrete procedures in arriving at answers, which in themselves make for better understanding. (3) The "proof method" is an excellent means by which to get children to correct their own errors. (4) The teacher or parent may check the child's understanding of a problem or a process by requiring him to prove his answer by using a diagram or drawing. (5) The requirements of "proof" of an answer to an arithmetical problem shows the advantage of the shorter and more abstract methods of working problems.

The last advantage listed should be noted seriously because the advocates of the "discovery method" for the teaching of arithmetic do not suggest "proof" by concrete and semi-concrete procedures in order to have less emphasis on work with the abstract symbols. "Proof" as presented in this discussion is advocated as a means to help the child better understand processes and operations indicated by the abstract symbols.

The types of "proof" which have been presented may well be employed by all teachers of arithmetic regardless of the general instructional method used in teaching the subject.

EDITOR'S NOTE. Dr. Petty wants pupils to understand the meaning and significance of their work with numbers and he also wants them to show "proof" that they understand. Perhaps "exidence" would be a better word than "proof" for his concept. However children seem to delight in showing and "proving" that they know what they are doing. A serious question arises. Should all pupils try to understand and to explain all of the operations of arithmetic or should we be content with explanations of only the simpler or more fundamental operations? Should we expect children of all mental levels to give explanations but which may be different for different levels of insight and ability? Is it true as we now suppose that the child who understands has the ability to project his learning through understanding into related areas or into similar circumstances and that perhaps also he retains his learning longer than does the one who merely learns by rote repetition of the thinking of others? We need some serious research on several kinds or modes of learning and the implications thereof in terms of the kind of arithmetic we want children to learn.

SOME HUNTERS (probably statistically minded) reported they had shot ducks and deer that had 25 heads and 60 feet. How many of each did they shoot?

WHAT IS THE DIFFERENCE between six dozen dozen and a half dozen dozen? Which is more valuable, half a silver dollar or a silver half dollar? Which is heavier, a pound of gold or a pound of feathers? Which is more, a quart of berries or a quart of vinegar? A year is often divided into quarters of three months each, which quarter is shortest?

IF, ON THE AVERAGE, a hen and a half lays an egg and a half in a day and a half, how many eggs will a dozen hens lay in twelve days?

I Went to an Arithmetic Workshop

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IN SUMMARIZING MY REACTIONS to the arithmetic workshop I would begin by saying that I was getting a new point of view, a changed outlook on the whole field of arithmetic. I would go on to explain that this had involved first of all a sort of "re-view" or backward glance at my past experiences with this subject; then a kind of bird's-eye view of the area of mathematics, followed by something like a 3-D view of the meaning of arithmetic as it confronts the child at school, and a preview of the possibilities regarding my future work in the classroom.

The "re-view" had brought me the rather startling realization that up to now arithmetic had been for me just a set of convenient rules and procedures, and that in the classroom I had been concerned mostly with passing on to my children memorized skills that I had found to be valuable, taking it more or less for granted that they were also learning to apply them.

The bird's eye view, I would explain, referred to my seeing, for the first time, the Hindu-Arabic number system as a marvelous unified whole, a complex yet in essence surprisingly simple and logical structure built on the concepts of number symbols, a base of ten, and the three fundamental ideas of combining, separating and comparing elements in quantitative situations.

Behind the Paper and Pencil

This view had led to the startling realization that I had never before seen, behind all the paper-and-pencil arithmetic I had done, any indication of the vast realm of meaningful experiences with numbers, accumulated by the race during hundreds of thousands of years and condensed and organized into our unique number system. This "meaning-approach" had constituted a sort of 3-D view for me, making the familiar number-facts suddenly stand out, as it were, with a new reality. Immediately I had decided to look for ways in which I could translate this realization into my work in the classroom.

It now seemed strange indeed that I had never marveled at the fact that with ten number symbols it was possible to write numbers that denoted distances of astronomical proportions. The part played by positional value, and by the number ten as the base from which one started over and over again, had indeed been known to me in so far as it entered into the rules and procedures with which I was familiar. But the "wonder" of this fact, and the absorbing interest it could hold for a child, and the immense value a realization of this fact could have in the early stages of number work, these were facets of the subject entirely new to me and they now presented marvelous possibilities.

With the new view of the meaning of number symbols and particularly of ten as base had come the recognition of the paramount importance of counting in the early number experiences of children, counting on and counting back, counting up and up on a number ladder on which every tenth step was of particular importance, and then down again. Imperceptibly one then got into the process of addition and subtraction as perfectly natural "shortcuts" in counting. Multiplication, too, could come later as *the* short cut *par excellence,* followed by the inverse process of taking the product apart in division.

The intimate interrelationships existing among the fundamental operations in the field of arithmetic were another facet of

the subject that, while of course known to me in the functional sense, had been completely "unrealized." What possibilities they presented for fascinating experiences in the classroom! And the simple postulates—simple yet tremendously significant—that underlie the Hindu-Arabic number system—they too were an asset, if properly understood, on which one could capitalize *ad infinitum*. If addends, minuends, subtrahends, multiplicands, and multipliers could be separated into any number of smaller parts, and the required operations performed on these, and the partial answers put together in any order, then how simple many computations could be made to be! Why, one could even do them in one's head! That was much more interesting than figuring them out with pencil and paper. They could be done in so many different ways. One could put these ways down and compare them to see which was the best way, the clearest, perhaps; or the simplest, or the shortest.

As the implications of these postulates had penetrated more and more into my thinking, I had resolved to do much more in the way of mental computation in the classroom than I had ever done before with my children. I now saw the tremendous value, too, of encouraging estimated answers, starting even with patently absurd guesses and refining them to closer estimates; testing them by thinking that made sense; doing many, many examples based on experiences that were real to children of the Third Grade and developing in them the habit of "thinking through" quantitative aspects to reach reasonable approximations and answers that had meaning. The records of their thinking, put on the board, could then be gradually matched up then to simpler forms till finally the standard algorisms were reached. And all this could be fun!

Insight and Thinking

Members of the workshop found it quite thrilling to arrive at answers by this process. It had been a real challenge to discover for ourselves the reasons behind various mathematical facts and rules—why, for instance, a multiplier always had to be abstract, and why a zero divisor constituted the one great exception in the logical structure of our number system. At times the period of confusion preceding learning had seemed rather long, as when we were trying to separate in our minds the two concepts involved in division—partitioning a dividend into a given number of equal parts, and comparing or measuring off a dividend in terms of a given divisor of the same denomination. But in the relaxed atmosphere of the classroom our confusion had become a source of merriment; and by skillful questioning we had been led to resolve our difficulties without undue embarrassment. In the process we had again seen demonstrated how there are many different ways of "thinking through" a problem to which one has never learned the answer, and many good algorisms or ways of recording the thinking in written form. The important thing was that the "thinking out" was done and that it was guided till it became clear, straight thinking.

This was to me the most fundamental insight gained by me this summer with regard to number work in my own classroom. I now intend to encourage my Third Graders to approach all quantitative situations in their experience, in the classroom and outside, from the "thinking out" angle. Hundreds of problems that have meaning for them can be worked out by us in the classroom, the majority of them by mental computation. Often the same problem can be developed by slight alterations into many similar examples, varied to suit the different levels of maturity evidenced by pupils, so that *all* will be involved and challenged. I venture to hope that my next year's class will end the school year with considerably more understanding of the meaning of quantitative situations and greater power in thinking their way through them than the class I said goodbye to in June.

Can $2+2=11$?

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For many millions of people the sum of 2+2 has always been 4. When a statement like this is made there arise many questions in the minds of even the least curious. Questions such as these: If two plus two isn't four what can it be? How can it be anything else? Can you show that it can be otherwise? Do you have to be a "mathemagician" to show that 2+2 can be other than 4? The answers rest in the solemn thought that 2+2 is 4 because of the viewpoint taken. Without any twisting of figures, or without any magic, it can very easily be shown that 2+2 can be 11 as well as 4. By the best rules $2+2=4$ is only a matter of viewpoint. An analogy suggests itself. If we take a trip to an amusement park and cast a glance at a distortion mirror we don't see ourselves as we think ourselves to be. Instead our positions might be reversed and we might find ourselves tall and skinny or short and pudgy when all the time we have been thinking otherwise. In other words we have received just another viewpoint on an old subject.

Now going back to $2+2=4$. This has been our old viewpoint when 10 is used as the base of our number system. There is a probability that we use base 10 because our ten fingers were very handy and could be put into practical use in the process of counting. As a matter of brief review a statement will be made concerning base 10. The figures used in base 10 are familiar to all of us. They are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. Then in order to count beyond the 9 we begin the process of prefixing the digit 1 as follows: 10, 11, 12, 13, 14, 15, 16, 17, 18, 19. After passing 19, we begin anew by prefixing the digit 2, as follows: 20, 21, · · · , 29 etc. The question on just how the process of addition is performed presents itself. For example, when we add (base 10) this generally is the procedure that we follow even though we don't realize these computations are taking place since many of the addition facts have been memorized:

$$\begin{array}{r} {\scriptstyle 011} \\ 384 \\ \underline{259} \\ 643 \end{array}$$

$4+9=13$; divide the 13 by your base (10 in this case) which gives 1 (to carry) and a remainder of 3. This residue (3) is brought down under the $9+4$. In the tens column we add $1+8+5=14$; divide the 14 by your base (10) which gives 1 (to carry) and a residue of 4. This 4 is brough down under the $8+5$. Proceeding to the hundred column we have $1+3+2=6$; divide the 6 by the base (10) which goes 0 times with a residue of 6. The six is brought down under the $3+2$ and the 0 is carried. The zero drops due to the fact there isn't anything to add it to.

Going back to $2+2=$ something other than 4, take 11 for example. Can it be possible that $2+2=11$? Yes, it is possible and just as feasible as $2+2=4$. To better understand why $2+2=11$, it will be necessary to make a few observations or to take another viewpoint of bases in reference to number systems. In base 3, for example, we are acquainted only with the digits 0, 1, 2. Remember in base 10, we were acquainted only with the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. Therefore to count in base 3 we would proceed as follows:

0, 1, 2
10, 11, 12
20, 21, 22
100, 101, 102 etc.

The same procedure in addition in base 3 (or any other base) is that as followed above for base 10. Therefore to add (base 3)

$$\begin{array}{r} 1 \\ 2 \\ +\ 2 \\ \hline 11 \end{array}$$

When we add 2+2 we get 4. As indicated in base 10 above we divide the 4 by our base (3 in this case) which gives 1 (to carry) and a residue of one which is brought down under the 2+2. In the second column we have only the 1 which is brought down in the answer. Therefore, depending on the viewpoint taken, 2+2 =4 (base 10) or 2+2=11 (base 3). There are two handy methods of checking your results that might be of interest. First, to show that 2+2=4 (base 10) set down the digits and by simply marking off

0, $\boxed{1, 2,}$ $\boxed{3, 4,}$ 5, 6, 7, 8, 9 etc.

By counting the first *two* numbers and then *two* more we arrive at our answer 4 (base 10). Likewise, with 2+2=11 (base 3). Proceed by setting down a few of the digits encountered in base 3 as follows:

0, $\boxed{1, \quad 2}$
$\boxed{10, 11,}$ 12
20, 21, 22, etc.

Then, as previously observed in base 10, we mark off two digits and follow this by marking off two more digits. Therefore the result 2+2=11 (base 3).

Another method used in checking might be noted. The problems here involved would be to show (a) that 11(base 3) is the same as 4(base 10) and (b) that 4(base 10) is identical to 11(base 3).

Taking problem (a) 11(base 3) = ? Going back for review in numeration in base 10, for an instant, you recall that if we have a number, for example 324, it is enumerated as follows:

3 (hundreds) 2 (tens) 4 (units)
which means

$$\begin{array}{lrl} 4\times & 1 \text{ (units)} & = \quad 4 \\ +2\times & 10 \text{ (base)} & = \quad 20 \\ +3\times & 100 \text{ (base squared)} & = \underline{300} \\ & & \quad\ \ 324 \end{array}$$

The same procedure is followed in 11(base 3). We don't speak of tens, hundred, etc. in base 3 for this terminology is reserved for base 10 only. Instead we proceed to enumerate as follows with the number 11(base 3):

$$\begin{array}{lr} 1\times 1 \text{ (units)} & =1 \\ +1\times 3 \text{ (base)} & =\underline{3} \\ & 4 \end{array}$$

Therefore in changing 11(base 3) to the ten's scale we find it to be 4(base 10).

The method used to change 4(base 10) to another base, say base 3, is a process of division. The procedure is as follows:

Change 4(base 10) to base 3. Set down the number (base 10) that is to be changed. This number is to be divided by the new base (3 in this case). For example: 4÷3=1 (three) and 1 (unit) remaining. In the number 11, on the base 3, the left digit *1* indicates a value of 3 (the base) and the right digit *1* indicates a value of 1 or a total of 4 as we think of this in terms of our ordinary base 10 numbers and hence 11(base 3)=4(base 10).

The procedure outlined above can be used in changing from one base to another or for addition in different scales. A similar procedure is followed in working with the other fundamental operations in arithmetic.

The value of a clear understanding of different bases is two-fold. First, by understanding thoroughly the possibilities involved in operating in other scales the teacher receives a better understanding of the intricacies involved in our own base 10. Secondly, the teacher can better understand the difficulties involved in addition (and the other fundamental operations) as encountered by young children.

An Approach to Per Cents

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IN MOST SCHOOLS we teach fractions, decimals and percentage in this order. It is a logical sequence leading from the general case to a more restrictive case.

Many times these very names will cause trembling—partly from fear and partly from the thrill of some new experience.

After reasonable skills are developed in manipulating fractions we can show that the decimals are a hand-picked group of fractions. Since decimals are a selected group, we should expect special properties and a refinement in the rules of operation.

Given time and writing surface, it would be possible to write out all possible fractions in an array as shown:

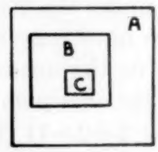
$$\begin{array}{r}
1/2,\ 1/3,\ 1/4 \cdots 1/10 \cdots 1/100 \cdots 1/1000 \text{ etc.} \\
2/3,\ 2/4 \cdots 2/10 \cdots 2/100 \cdots 2/1000 \cdots \\
3/4 \cdots 3/10 \cdots 3/100 \cdots 3/1000 \cdots \\
4/5 \cdots 4/10 \cdots 4/100 \cdots 4/1000 \cdots
\end{array}$$

If we now pick from this array only those fractions whose denominators are a power of 10. 10^n, $n=1, 2, 3, \cdots$, we have the decimal group, $1/10^n$.

We can show the relationship between these fractions and the decimal notation. $1/10^n$, when $n=1$, $1/10=.1$; when $n=2$, $1/100=.01$, etc. This is a new way for writing the special grouping of common fractions—namely the decimals.

A further selection can be made from the fraction array or the decimal group. This time only one classification—the hundredths where $n=2$. We give this exclusive group a special name—"per cent." Per cent means hundredths and we give it a special symbol, or badge of office—"%." Thus $15/100=.15=15\%$.

We can represent the entire field of fractions with a large square A, then decimals could be grouped within a smaller square B, and per cents, being a part of the decimals, can be represented by square C. Thus every per cent is a decimal and every decimal is a fraction as shown.



We can represent the entire student body as group A. Those that are inclined toward playing on school teams in group B, while those in group C are on the varsity teams. Thus every boy in group C is interested in sports and is a member of the student body.

It is well to describe the symbol % to read per cent and to mean hundredths. Thus we can read 17% as 17 hundredths —.17 or 17/100 when solving problems. Our pupils know many symbols in other fields as in music, highway signs, scouting. This sign, % is another they should know and understand how to use.

EDITOR'S NOTE. At what grade level is Mr. Hibbard's presentation most suitable? Is it useful in the junior high school after pupils have developed an understanding of exponents? His pictorial scheme tends to suggest that decimal fractions are fewer in number than common fractions and that per cents are still fewer. Is this correct or is restrictive in terms of the analysis he has proposed? Can any decimal be expressed as a per cent and vice versa?

Learning Arithmetic from Kindergarten to Grade 6

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AFTER I STARTED to go to school, my life seemed to be the most unpleasant one. I had a very moody and terrible teacher. In my point of view, she never had good temper. She taught me arithmetic and everything else.

As far as I can recall, my first contact with the number system was to learn how to write the numbers correctly from one to ten, without knowing what the symbols represented, just to try to copy the figures given by the teacher on the blackboard.

The next step, when learning the operations of addition, subtraction, multiplication, and division, was to try to memorize exactly the way in which the teacher did them. I was not taught to know what the ones, the tens, and the hundreds really are. I just worked out the problems mechanically.

The only concrete materials known to me in those days were just fingers and toes. When the operation of multiplication was introduced to our class, we were trained to be parrots. Every afternoon before going home we had to recite the tables of multiplication from two to twelve without any understanding.

The method of teaching arithmetic in those days was drill without understanding. I don't think that understanding was recognized as an important part of learning by any teachers, even the slightest bit.

We learned arithmetic mechanically through punishment. When we missed any problems, we were beaten on the hand. The number of beats depended on the number of wrong answers.

I began to enjoy learning arithmetic when I was in grade three, because I had a very kind and nice teacher. Later on I hated arithmetic again because I had a terrible teacher once more. The very thing I always remember is that whenever the teacher finished her explanation, she asked the class to raise some questions and once I did. Instead of getting her helpful answer, she said that I did not understand and made an ugly face at me. I remember that I was very furious at her and since then I hated both the subject and the teacher.

From my point of view, the attitudes of teachers have very much influence on the result of learning of children, especially very young children. The more the teachers gain favor in the pupil's minds, the more they are successful in teaching.

Certainly, the method of teaching goes along with the teacher's attitude. The method used by most teachers in those days, in my country, was the method of wasting time: instead of trying to teach us to understand, they tried to force us to memorize and they spent a lot of time to check our memory, one by one.

I never felt the atmosphere of freedom in my classroom but only that of punishment.

EDITOR'S NOTE. When Professor Fehr of Teachers College, Columbia University received Miss Ratanakul's statement of experience in learning arithmetic, he prepared the following comment.

This is the story of a teacher from Thailand, but it could well be the story of teachers from every country in the world. The teaching of arithmetic, to more than 70% of all present elementary teachers in the United States, by their own report has been:

> They did not understand what they were doing.
> There was no reliance or premium placed on thinking.
> The teaching was arbitrary; rote dictations, drill.

Arithmetic was copying number symbols; saying words in order; rote memory of facts; flash cards ad infinitum; borrowing and paying back; putting, placing; speed and more speed; tests and tests and tests; impossible word problems; a subject of punishment.

Will our future citizens say the same of our present teaching? Let us hope not.

HOWARD F. FEHR